Linear Algebra Week 1

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1 Vectors in $\mathbb{R}^n$

Vectors are ordered lists of numbers (i.e. the position/index of each entry is fixed) written as

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$$

Often it makes sense to consider a vector's geometric representation in a coordinate system: Each entry corresponds to one direction. We draw an arrow from the origin to the coordinate the vector specifies:

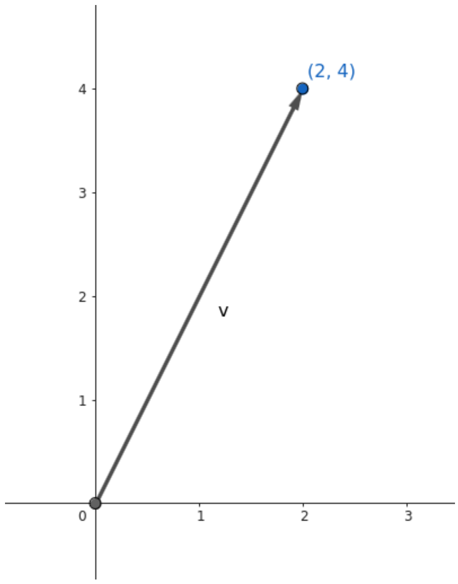

Figure 1: The vector $v = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ where 2 is the x and 4 the y coordinate

We can add two vectors and multiply a vector with a scalar (real number that scales the vector):

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} v_1 + u_1 \\ \vdots \\ v_n + u_n \end{bmatrix}$$

$$\alpha \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \alpha v_1 \\ \vdots \\ \alpha v_n \end{bmatrix}$$

Vector addition corresponds to chaining two vectors together: place one starting from the origin and the other starting from the tip of the first vector. Multiplying a vector with a scalar scales that vector: We can make it longer, shorter or flip its orientation by multiplying with a negative number.

2 Linear combinations of vectors, span

Definition A *linear combination* of a sequence of vectors $v_1, \dots, v_k \in \mathbb{R}^n$ is a vector

$$\alpha_1 v_1 + \cdots + \alpha_k v_k$$

where $\alpha_1, \dots, \alpha_k \in \mathbb{R}$

A linear combination is a vector that results from combining vectors of a sequence of vectors (adding them), each of them multiplied by a scalar.

Example

$$\begin{bmatrix} 9 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Definition The *span* of a sequence of vectors $v_1, \dots, v_k \in \mathbb{R}^n$ is the set of all linear combinations of that sequence:

$$\mathsf{span}(v_1, \dots, v_k) = \{\alpha_1 v_1 + \cdots + \alpha_k v_k \in \mathbb{R}^n \mid \alpha_1, \dots, \alpha_k \in \mathbb{R}\}$$

Looking at the set of all possible linear combinations is often very useful: We can see which geometric form a sequence of vectors fills and whether or not it is possible to express some vector as a linear combinations of others.

3 Linear systems of equations

An example of a linear system of equations with $m = 2$ equations and $n = 2$ unknowns is:

$$\begin{aligned} 4x - y &= -1 \\ -x - y &= -6 \end{aligned}$$

There are two ways to interpret this system of equations geometrically: the *row picture* and *column picture*.

3.1 Row picture

The *row picture* is probably familiar to many from high school: We can see each *row* of the system as a linear function and sketch it in a coordinate system.
If the two functions intersect in one point, that point is the solution to the system of equations. If both functions are the same, we get an infinite amount of solutions - all that lie on the line defined by the function.
Lastly, if the two lines don't meet (e.g. because they are parallel to each other) no solution exists.

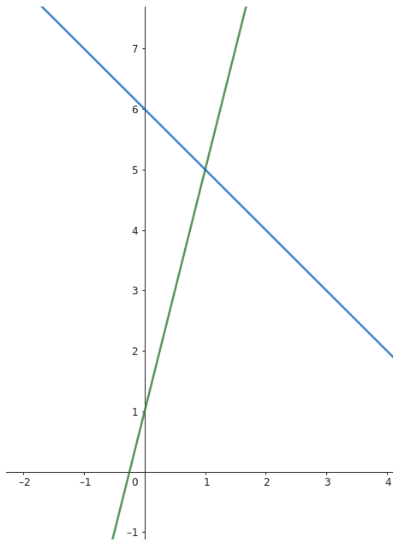

Figure 2: The *row picture* of the system above

3.2 Column picture

The *column picture* is the linear algebra way to look at a system of linear equations: Each column of the system is a vector and the system is solved by finding a linear combination of those columns (e.g. the right values x, y to make both equations above true at the same time).

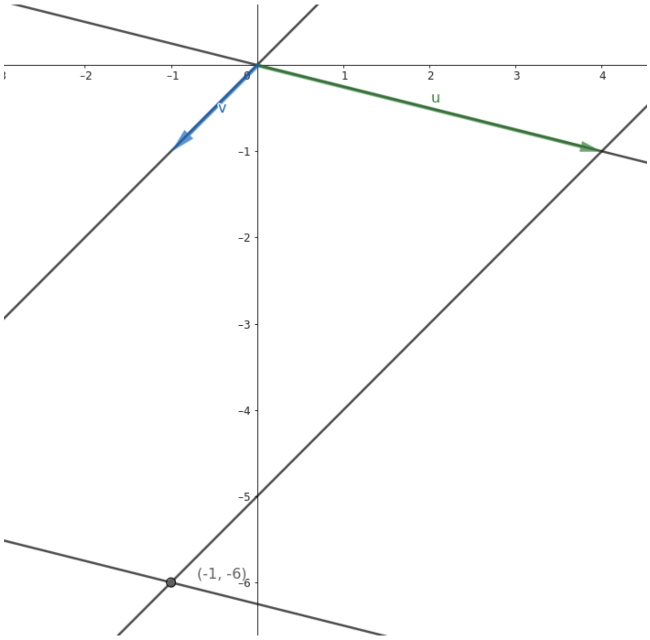

Figure 3: The *column picture* of the system above

From this picture we can also read off the solution: We see that vector u is needed once and vector v five times such that their linear combination is the vector $\begin{bmatrix} -1 \\ -6 \end{bmatrix}$.
Writing this in vector form we have:

$$\alpha_1 \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \end{bmatrix}$$

4 Vector matrix multiplication

Another way to write the linear combination from the above *column picture* is in matrix form (we call this matrix *coefficient matrix* as the matrix entries are the coefficients of the unknowns in the system of equations):

$$\alpha_1 \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \end{bmatrix}$$

5 Exercise: linear combination that equals 0

Find a linear combination that equals the zero vector where not all scalars are zero:

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + \alpha_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Is this sequence of vectors linearly independent/dependent? (see week 2 for definition)

Source of exercise: Chapter 1.3, Introduction to Linear Algebra: Strang