Linear Algebra
Week 12

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G-04

Quiz

Let $A, B \in \mathbb{R}^{n\times n}$ and $W \in \mathbb{R}^{m\times n}$:

1. How is $\det(A)$ related to $\det(7A)$? $\det(7A) = 7^n \det(A)$
2. The determinant is only defined for square matrices TRUE
3. If two rows or columns of A are identical, $\det(A) = 0$ TRUE, A is not invertible
4. Applying elimination matrices on A doesn't change $\det(A)$ FALSE
   → may change sign and if we include multiplying row by scalar, also by scalar
5. $\det(A) = -\det(A^T)$ FALSE, $\det A = \det A^T$ (see next page)
6. $\det(AB) = \det(B)\det(A)$ TRUE (see next page)
7. $\det(A^2) = \det(A)\det(A)$ TRUE (6. applied on $AB = AA$)
8. $\det(A^{-1}) = \frac{1}{\det(A)}$ TRUE $1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$
   $\Rightarrow \det(A^{-1}) = \frac{1}{\det A}$
9. $WW^+ = I$ or $W^+W = I$ FALSE, only if full column or row rank
   → see week 10, $x^+ = W^+b$ still has nice properties!
10. If W has full column rank, W^TW has full row rank and is surjective TRUE
    → see week 10, W^TW has full rank and is invertible
11. If W has full row rank, $m \geq n$ and W is surjective FALSE, m may be smaller than n

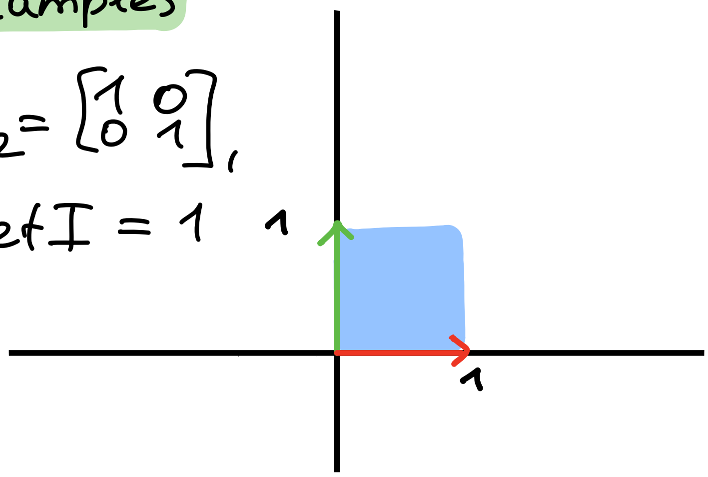
The determinant

We consider $\det A = D(a_1, \dots, a_n)$ where $a_1, \dots, a_n \in \mathbb{R}^n$ are the columns of $A \in \mathbb{R}^{n\times n}$ as a function $\mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$ that gives us the oriented volume of the n-dimensional parallelogram spaned by $a_1, \dots, a_n$.

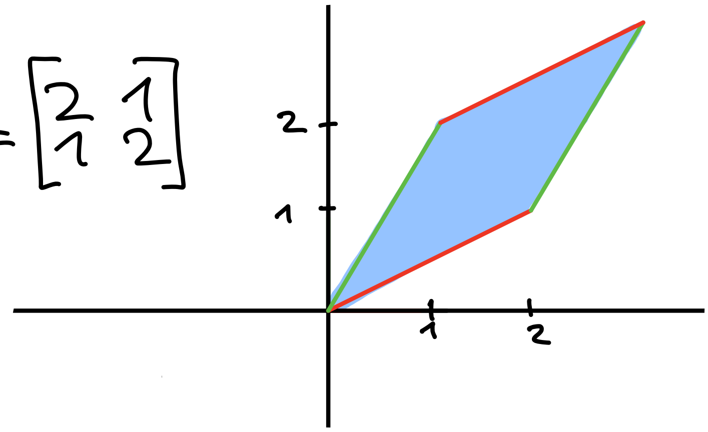
Examples

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$\det I = 1$



$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$



Most important properties

1) Linear in each column/row
2) Swapping columns/rows changes sign
3) $\det A$ doesn't change if we add multiple of one column (row) to another
4) $\det A \neq 0 \Longleftrightarrow A$ is invertible
5) $\det A = \det A^T$
6) $\det(AB) = \det(A)\det(B)$
7) $\det(\alpha A) = \alpha^n \det(A)$ $\longrightarrow$ n times linearity per column/row
8) determinant of triangular matrix is product of diagonal entries
   ↘ in particular of diagonal matrix
9) $\det\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = \det A \det D$

Deriving the formal definition

$$D(v_1, \dots, v_n) \qquad A = \begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix}$$

$$= D\Big(\sum_{j=1}^{n} a_{j,1} e_j, v_2, \dots, v_n\Big) \qquad \Big(v_1 = \sum_{j=1}^{n} a_{j,1} e_j\Big)$$

$$= \sum_{j=1} a_{j,1} \, D(e_j, v_2, \dots, v_n) \qquad \text{(linearity in each column/row)}$$

$\vdots$ (repeating the step above for $v_2, \dots, v_n$)

$$= \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_n=1}^{n} a_{j_1,1} a_{j_2,2} \cdots a_{j_n,n} D(e_{j_1}, e_{j_2}, \dots, e_{j_n})$$

this sum is gigantic with n^n terms

BUT: $D(e_{j_1}, \dots, e_{j_n}) = 0$ if two columns are identical, meaning two of the indices $j_1, \dots, j_n$ are the same.
The only sequences of indices $j_1, \dots, j_n$ that remain are rearrangements of $\{1, \dots, n\}$: so called permutations.
We consider one such permutation as a function
$\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$
where $\sigma(1), \dots, \sigma(n)$ is the new order/arrangement.

form group S_n of order $n!$

We now have:

$$= \sum_{\sigma \in S_n} a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n} \underbrace{D(e_{\sigma(1)}, \dots, e_{\sigma(n)})}_{\substack{=1 \text{ or } -1 \text{ depending} \\ \text{on number of row swaps} \\ \text{(odd/even) from } I}}$$

We define

$$\text{sign}(\sigma) = \begin{cases} 1 & \text{even } \# \text{ swaps required from } \{1, \dots, n\} \text{ to } \{\sigma(1), \dots, \sigma(n)\} \\ -1 & \text{odd} \end{cases}$$

to get $\det A = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i} = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$

order doesn't matter

The above definition can be used to find formulas for the determinant of 2×2 or 3×3 matrices.
However: Don't use it directly for computations!
The following page has two methods for computing $\det A$.

Example There are two permutations of $1, 2$:

- $\sigma_1(1) = 1, \quad \sigma_1(2) = 2 \implies \operatorname{sign}(\sigma_1) = 1$
- $\sigma_2(1) = 2, \quad \sigma_2(2) = 1 \implies \operatorname{sign}(\sigma_2) = 1$

For a 2×2 matrix this gives us:

$$\det A = \operatorname{sign}(\sigma_1)\, a_{1,\sigma_1(1)}\, a_{2,\sigma_1(2)} + \operatorname{sign}(\sigma_2)\, a_{1,\sigma_2(1)}\, a_{2,\sigma_2(2)}$$

$$= a_{11} a_{22} - a_{12} a_{21}$$

Computing the determinant

1) **Laplace-Expansion** $\longrightarrow$ benefits from rows / columns that have many zeros

expand

k^{th} row: $$\det A = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det A[k,j]$$

ℓ^{th} column: $$\det A = \sum_{j=1}^{n} a_{j\ell} \underbrace{(-1)^{j+\ell} \det A[j,\ell]}_{\text{cofactor } j\ell,\ C_{j\ell}}$$

General formula for 2x2 matrices:

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Expansion along first row:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ 32 & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

2) **Gauss-Elimination** $\longrightarrow$ used in practice $O(n^3)$

$$\det A = (-1)^{\underset{= \#\text{row swaps}}{r}} \prod_{i=1}^{n} r_{ii}$$

Product of pivots in REF(A)

• Note: don't multiply row by scalar! This might change $\det A$!

There are methods for computing A^{-1} and $x = A^{-1}b$ using the determinant. You can find them in the script.

Example

$$\det\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & -5 \\ 6 & -4 & 3 \end{bmatrix} = 0 \cdot \det\begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix} - 1 \cdot \det\begin{bmatrix} 1 & 1 \\ -4 & 3 \end{bmatrix} + 6 \cdot \det\begin{bmatrix} 1 & 1 \\ 2 & -5 \end{bmatrix}$$

$$= -1\,(3+4) + 6 \cdot (-5-2)$$

$$= -7 + 6(-7) = -49$$

Complex Numbers

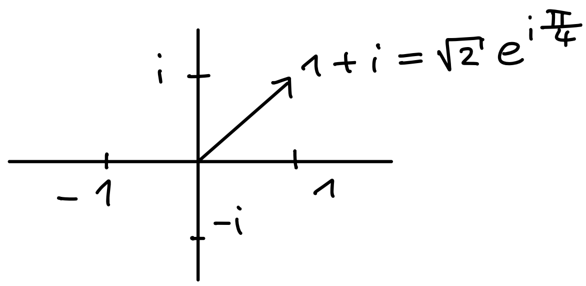
Any $z \in \mathbb{C}$ has the form $z = a + bi$ where $i^2 = -1$.

a is the real part of z: $\mathrm{Re}(z) = a$
b is the imaginary part of z: $\mathrm{Im}(z) = b$

$\bar{z} = \overline{(a+bi)} = a - bi$ is the complex conjugate of z

$|z| = \sqrt{a^2+b^2} = \sqrt{z\,\bar{z}}$ is the modulus of z → corresponds to length in complex plane

We can see z as a vector in the complex plane
→ x axis is the real axis, y axis the imaginary axis



Any complex number z has polar form $re^{i\theta}$:

- $z = a + bi = r(\cos\theta + i\sin\theta) = re^{i\theta}$

  → $r = |z|$
  → $a = r\cos\theta$
  → $b = r\sin\theta$

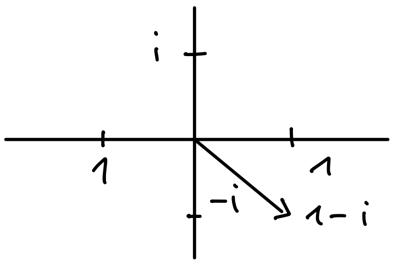
- Multiplying by $re^{i\theta}$ has effect of stretching by r and rotating counterclockwise by θ in the complex plane
  example: multiply by $i = e^{i\frac{\pi}{2}}$ ⟷ rotate by 90 degrees ($= \frac{\pi}{2}$ radians)

$A^* = \overline{A^T}$ is the conjugate transpose of A
(often also called hermitian transpose)

→ Quickly estimate/compute values of sin/cos utilizing the unit circle

Example For $z = 1 - i$, find $\bar{z}$ and $r = |z|$, θ such that
$z = re^{i\theta}$:

$$\bar{z} = \overline{(1-i)} = 1 + i$$
$$r = |z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$



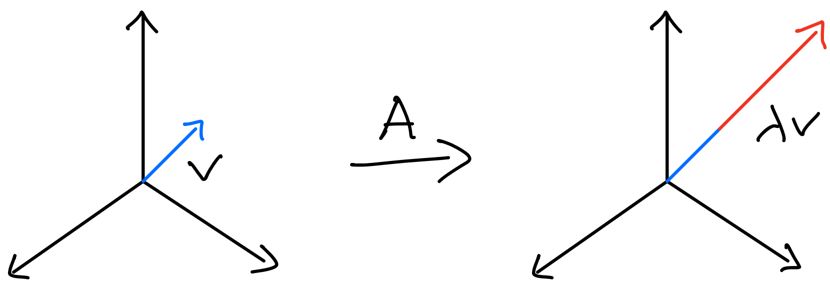
We see that this is a 45° rotation clockwise which is $-\frac{\pi}{4} = 2\pi - \frac{\pi}{4} = \frac{7}{4}\pi$ radians
(we could also use $1 - i = \sqrt{2}(\cos\theta + i\sin\theta)$)

Eigenvalues and eigenvectors

Let $A \in \mathbb{R}^{n \times n}$. We call nonzero $v \in \mathbb{C}^n$ eigenvector of A correspondonding to the eigenvalue $\lambda \in \mathbb{C}$ if

$$Av = \lambda v$$

Geometrically, v is only stretched but did not change direction after applying A:



$$\begin{aligned} & Av = \lambda v \\ \iff\ & Av - \lambda v = 0 \\ \iff\ & (A - \lambda I)v = 0 \\ \iff\ & \det(A - \lambda I) = 0 \text{ and } v \in \underbrace{N(A - \lambda I)}_{\text{all eigenvectors corresponding to } \lambda} \end{aligned}$$

$\det(A - \lambda I) = 0$ $\downarrow$ gives us eigenvalues

characteristic polynomial $\chi_A(\lambda) = \det(A - \lambda I)$

↳ why is this a polynomial of degree n?
→ consider formal definition of determinant

algebraic multiplicity of λ . . . multiplicity of λ as root of $\chi(A)$

geometric multiplicity of λ . . . $\dim N(A - \lambda I)$

"how many dimensions of eigenvectors corresponding to λ"

$$\chi_A(\lambda) = (-\lambda)^n + Tr(A)\,(-\lambda)^{n-1} + \dots + \det A$$

Fundamental theorem of algebra:

$$\chi_A(\lambda) = (\lambda_1 - \lambda) \cdot \ldots \cdot (\lambda_n - \lambda)$$
$$= (\lambda_1 - \lambda)^{p_1} \cdot \ldots \cdot (\lambda_k - \lambda)^{p_k}$$

algebraic multiplicity

$\chi_A(\lambda) = (2-\lambda)^3(1-\lambda) \longrightarrow$ 2 is eigenvalue with algebraic multiplicity 3, 1 with algebraic multiplicity 1

Algorithm for computing eigenvalues/eigenvectors

1. Find $\chi_A(\lambda) = \det(A - \lambda I)$
2. Find solutions to $\chi_A(\lambda) = 0$
   $\longrightarrow$ eigenvalues λ
3. For every unique λ, find basis of $N(A - \lambda I)$ with elimination
   $\longrightarrow$ eigenvectors v

Example Find all eigenvalues and eigenvectors of
$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

1. $\chi_A(\lambda) = \det \begin{bmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix} = (4-\lambda)(-3-\lambda) + 10$
$= -12 - 4\lambda + 3\lambda + \lambda^2 + 10 = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$
2. $\chi_A(\lambda) = (\lambda+1)(\lambda-2) = 0 \iff \lambda = -1$ or $\lambda = 2$
3. (i) $\lambda = -1$, we solve $(A+I)v = 0$ for $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$:

$$\begin{array}{c} \\ \frac{2}{5} \end{array}\begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 5 & -5 \\ 0 & 0 \end{bmatrix} \Rightarrow$$

$5v_1 - 5v_2 = 0 \Rightarrow v_1 = v_2$, $v = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

v_2 is free

A basis for $N(A+I)$ is given by $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(ii) $\lambda = 2$

$$A - 2I = \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow 2v_1 - 5v_2 = 0 \Rightarrow v_1 = \frac{5}{2} v_2$

$\left\{ \begin{bmatrix} 5/2 \\ 1 \end{bmatrix} \right\}$ is a basis for $N(A-2I)$.

$\longrightarrow$ check by testing $Av = \lambda v$

References:

Last years course for some definitions

Sergey Treil, Linear Algebra Done Wrong, https://www.math.brown.edu/streil/papers/LADW/LADW_2021_01-11.pdf