Linear Algebra
Week 13

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G-04

Quiz

Let $A, Q \in \mathbb{C}^{n\times n}$ and $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A. The trace of A is defined as $\mathrm{Tr}(A) = \sum_{i=1}^{n} (A)_{ii}$, $A^* = \overline{A}^T$.

1. $\det(A^*) = \det(A)$ FALSE, $\det(A^*) = \det(\bar{A}^T) = \det(\bar{A}) = \overline{\det(A)}$
2. The eigenvalues of a triangular matrix are given by its diagonal entries. (Note: A diagonal matrix is triangular) for a diagonal matrix, $\det A = \prod_{i=1}^{n} a_{ii}$ $\Rightarrow \chi_A(\lambda) = \det(A - \lambda I) = \prod_{i=1}^{n} (a_{ii} - \lambda)$
3. $\sum_{i=1}^{n} \lambda_i = \mathrm{Tr}(A)$, $\prod_{i=1}^{n} \lambda_i = \det(A)$ TRUE, proven in lecture
4. Eigenvectors corresponding to different eigenvalues are not necessarily linearly independent FALSE, eigenvectors corresponding to different eigenvalues are always linearly independent
5. If we know all eigenvalues of A, we know if A is invertible. TRUE
   A invertible $\Leftrightarrow$ 0 no eigenvalue of A
6. What are the algebraic and geometric multiplicities of an eigenvalue λ?
   multiplicity of λ as root of $\det(A - \lambda I)$; $\dim N(A - \lambda I)$
7. By the fundamental theorem of algebra, any polynomial with real coefficients has real roots. FALSE, a polynomial with real coefficients has complex roots (not necessarily real)
8. $\det(PA) = \det(A)$ when P is a permutation matrix (attained from swapping columns/rows of the identity matrix) FALSE, $\det(PA) = \pm \det A$ depending on number of row/column swaps
9. $\det(Q)| = \pm 1$ and $|\lambda| = 1$ if $Q \in \mathbb{C}^{n\times n}$ is an orthogonal matrix ($Q^T Q = I$) with eigenvalue λ. (Extra question: what if Q is unitary, i.e. $Q^* Q = I$?) TRUE

We proof the two statements separately:

1) $|\lambda| = 1$

$$Qv = \lambda v \Rightarrow \|Qv\| = \|\lambda v\| = |\lambda| \, \|v\| \overset{Q \text{ preserves lengths}}{=} \|v\| \Rightarrow |\lambda| = 1$$

2) $\det Q = \pm 1$

$$1 = \det I = \det(Q^T Q) = \det(Q^T) \det(Q) = \det(Q)^2$$
$$\Rightarrow \det Q| = \pm 1$$

in the case that Q is unitary we get

$$1 = \det I = \det(Q^* Q) = \det(Q^*) \det(Q) = \overline{\det(Q)} \det(Q)$$
$$= |\det(Q)|^2 \Rightarrow |\det(Q)| = 1$$ (on unit circle in complex plane)

Change of basis

Consider two bases $B_1 = \{u_1, \dots, u_n\}$ of $\mathbb{R}^n$ → this works for any vector space, not just $\mathbb{R}^n$

$B_2 = \{v_1, \dots, v_n\}$

Any vector $x \in \mathbb{R}^n$ can be represented as a unique linear combination of vectors in B_1 (or B_2). The coefficients of this linear combination written as a vector results in the coordinate vector of x regarding B_1 (or B_2), $[X]_B$ (or $[X]_{B_2}$)

A change of basis matrix allows us to switch between coordinate representations regarding different bases.

$$T = \begin{bmatrix} | & & | \\ [v_1]_{B_1} & & [v_n]_{B_1} \\ | & & | \end{bmatrix}$$

is the change of basis matrix from B_1 to B_2

takes a vector in B_2 representation to B_1

this name is usually used: T transforms space in B_1 to space in B_2

$$T^{-1} = \begin{bmatrix} | & & | \\ [u_1]_{B_2} & & [u_n]_{B_2} \\ | & & | \end{bmatrix}$$

is the change of basis matrix from B_2 to B_1

takes a vector in B_1 representation to B_2

$$\begin{tikzcd}
B_1 \arrow[rrr, bend right, "{T^{-1}[x]_{B_1} = [X]_{B_2}}"'] & & & B_2 \arrow[lll, bend right, "{T[X]_{B_2} = [x]_{B_1}}"']
\end{tikzcd}$$

Example $B_1 = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$, $B_2 = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\3\end{bmatrix} \right\}$

The change of basis matrix from B_1 to B_2 is

$$T = \left[\begin{bmatrix}1\\0\\0\end{bmatrix}_{B_1} \begin{bmatrix}0\\2\\0\end{bmatrix}_{B_1} \begin{bmatrix}0\\0\\3\end{bmatrix}_{B_1} \right] = \begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$$

and takes a vector in coordinate representation regarding B_2 to coordinate representation regarding B_1

The change of basis matrix from B_2 to B_1 is

$$T^{-1} = \left[\begin{bmatrix}1\\0\\0\end{bmatrix}_{B_2} \begin{bmatrix}0\\1\\0\end{bmatrix}_{B_2} \begin{bmatrix}0\\0\\1\end{bmatrix}_{B_2} \right] = \begin{bmatrix}1&0&0\\0&1/2&0\\0&0&1/3\end{bmatrix}$$

and takes a vector in coordinate representation regarding B_1 to coordinate representation regarding B_2

Composition of change of basis matrices:

Let T_1 be the change of basis matrix from B_0 to B_1

T_2 from B_0 to B_2

$\Longrightarrow T_1^{-1} T_2$ is the change of basis matrix from B_1 to B_2

takes vector in B_0 to B_1 representation

takes vector from B_2 to B_0

We can use this to construct representation matrices regarding bases of our choice! Assume A is the matrix of a linear map in regard to the basis B_0 (e.g. standard basis):

Then $T_1^{-1} A T_2$ corresponds to the same linear map regarding bases B_1, B_2, taking as input a vector in B_2 and giving one back in coordinate representation regarding B_1.

Exercise Consider the linear map $L: \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$L(x,y,z)^T = (3x+4y, 2z, x+y+z)^T$

1) Find the representation matrix $A \in \mathbb{R}^{3\times 3}$ of L regarding the canonical basis $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^3$ (input / output in this basis)

$\to$ Transform basis vectors
$L(1,0,0)^T = (3,0,1)^T$, $L(0,1,0)^T = (4,0,1)^T$,
$L(0,0,1)^T = (0,2,1)^T$

$$\Rightarrow A = \begin{bmatrix} | & | & | \\ L(e_1) & L(e_2) & L(e_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

2) Consider the two bases $B_1 = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\3\end{bmatrix}\right\}$, $B_2 = \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$.

Find the change of basis matrices
T_1 from the standard basis to B_1, T_1^{-1} (from example)
T_2 from the standard basis to B_2
and then from B_1 to B_2 given by $T_1^{-1} T_2$

Finally, find the representation matrix of L in regard to bases B_1, B_2 that takes as input a vector in B_2 representation and outputs a vector in coordinates regarding B_1

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$T_1^{-1} A T_2$ is the representation matrix of L in regard to bases B_1, B_2

Similar matrices, eigendecomposition

$A, B \in \mathbb{R}^{n \times n}$ are called similar if $B = S^{-1} A S$ for some invertible matrix $S \in \mathbb{R}^{n \times n}$

→ A and B are the same linear map under different bases, S and S^{-1} are the respective change of basis matrices

Some properties

- Similar matrices have the same
  1) characteristic polynomial
  2) eigenvalues with same algebraic and geometric multiplicity — follows from (1)
  3) rank (multiplying with full rank matrix doesn't change rank)
  4) trace (1)
  5) determinant (1)

- The "similar" relation is an equivalence relation
  → in particular, transitive (see ex. 12.2)

A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable if it is similar to a diagonal matrix $\iff A = V \Lambda V^{-1}$ for some diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$. We give this decomposition a special name: eigenvalue decomposition.

Eigenvalue decomposition

$$A = \underbrace{V}_{\substack{\text{eigenvectors}\\ \text{as columns}}} \underbrace{\Lambda}_{\substack{\text{eigenvalues on diagonal,}\\ \text{other entries zero}}} V^{-1}$$

Note:
The matrix V has to be the left matrix

The following statements are equivalent: $A \in \mathbb{R}^{n \times n}$

- A is diagonalizable
- An eigenvalue decomposition of A exists
- There exists a basis of eigenvectors of A of $\mathbb{R}^n$
- There exist n linearly independent eigenvectors of A
- For every eigenvalue, the geometric and algebraic multiplicities are the same

Claim:
For any eigenvalue λ', $1 \leq$ geometric multiplicity $\leq$ algebraic multiplicity

Proof that geometric multiplicity $\leq$ algebraic multiplicity:
Assume λ' is an eigenvalue of $A \in \mathbb{R}^{n \times n}$ with geometric multiplicity k. $\dim N(A - \lambda' I) = k$, there exist k eigenvectors $v_1, \dots, v_k$ corresponding to λ'. We extend these vectors to a basis and get an invertible matrix V:

$$V = \begin{bmatrix} | & & | & | & & | \\ v_1 & \cdots & v_k & v_{k+1} & \cdots & v_n \\ | & & | & | & & | \end{bmatrix}$$

$$\qquad \underbrace{\qquad\qquad}_{\text{eigenvectors}} \underbrace{\qquad\qquad}_{\text{other vectors}}$$

$$AV = \begin{bmatrix} | & & | \\ Av_1 & \cdots & Av_n \\ | & & | \end{bmatrix} = \begin{bmatrix} \lambda' v_1 & 0 & 0 & 0 & | & & | \\ 0 & \lambda' v_2 & & \vdots & & & \\ 0 & 0 & \ddots & \lambda' v_k & Av_{k+1} & \cdots & Av_n \\ \vdots & \vdots & & \vdots & & & \\ 0 & 0 & 0 & 0 & | & & | \end{bmatrix}$$

We left multiply with V^{-1}: $(V^{-1}V = I \Rightarrow V^{-1}v_i = e_i)$

$$V^{-1}AV = \begin{bmatrix} | & & | & | & & | \\ V^{-1}\lambda' v_1 & \cdots & V^{-1}\lambda' v_k & V^{-1}Av_{k+1} & \cdots & V^{-1}Av_n \\ | & & | & | & & | \end{bmatrix} = \begin{bmatrix} \begin{matrix} \lambda' & 0 & 0 & 0 \\ 0 & \lambda' & \vdots & 0 \\ 0 & \cdots & & \lambda' \end{matrix} & B \\ 0 & C \end{bmatrix}$$

A and $V^{-1}AV$ are similar and thus have the same characteristic polynomial:

$$\det(A-\lambda I) = \det(V^{-1}AV - \lambda I) = \det \begin{bmatrix} (\lambda'-\lambda)I_{k\times k} & B \\ 0 & C-\lambda I_{k\times k} \end{bmatrix} =$$

$$= \det((\lambda'-\lambda)I_{k\times k}) \det(C-\lambda I_{k\times k}) = (\lambda'-\lambda)^k \det(C-\lambda I_{k\times k})$$

The algebraic multiplicity of λ' is at least $k = \dim N(A-\lambda' I)$, the geometric multiplicity of λ'.

Example Which of the following matrices is diagonalizable?

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ — A diagonal matrix is always diagonalizable. In this case, the eigenvalues are 1 and 2.

$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ — $\det(B-\lambda I) = \lambda^2 = 0 \Leftrightarrow \lambda = 0$

0 is an e.v. with algebraic mult. 2. However, B has rank 1, $N(B)$ dimension 1 meaning the geometric mult. of 0 is $1 \neq 2$

$\Rightarrow$ B is not diagonalizable

$C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ — not diagonalizable, analogous argument to B

Unfortunately, not every matrix is diagonalizable (as we see). However, for symmetric matrices the spectral theorem applies.

The Spectral Theorem

Let $A \in \mathbb{R}^{n\times n}$ be symmetric $(A^T = A)$.

- A has n real eigenvalues
- There exists an orthonormal basis of $\mathbb{R}^n$ of eigenvectors of A
- A has an eigendecomposition $A = U \Delta U^T$ where U is orthogonal (and U's columns form an orthonormal basis of $\mathbb{R}^n$)

Exercise 11.1.b

Find the representation matrix B of the reflection through the plane given by $P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; 3x + 4y = 0 \right\}$

The main idea here is that we can decompose $\mathbb{R}^3$ into $P + P^\perp = \mathbb{R}^3$ where $P = \text{span}\{n\}$ where n is a normal vector on P (length one and orthogonal to all vectors in P).

Projecting a vector x onto span$\{n\}$ gives us only the part of x that's orthogonal to P. Subtracting this projection once from x eliminates this orthogonal part completely, subtracting it twice flips the sign of the orthogonal part. We can express this as follows:

$$x = \underbrace{x_{\_}}_{\text{parallel to } P} + \underbrace{x_1}_{\text{orthogonal to } P}$$

The projection matrix onto span$\{n\}$ is $\dfrac{nn^T}{n^Tn} = nn^T$

$$nn^Tx = x_1 \implies x - 2nn^Tx = \underbrace{(I - 2nn^T)}_{\text{this is the matrix } B} x = x_{\_} - x_1$$

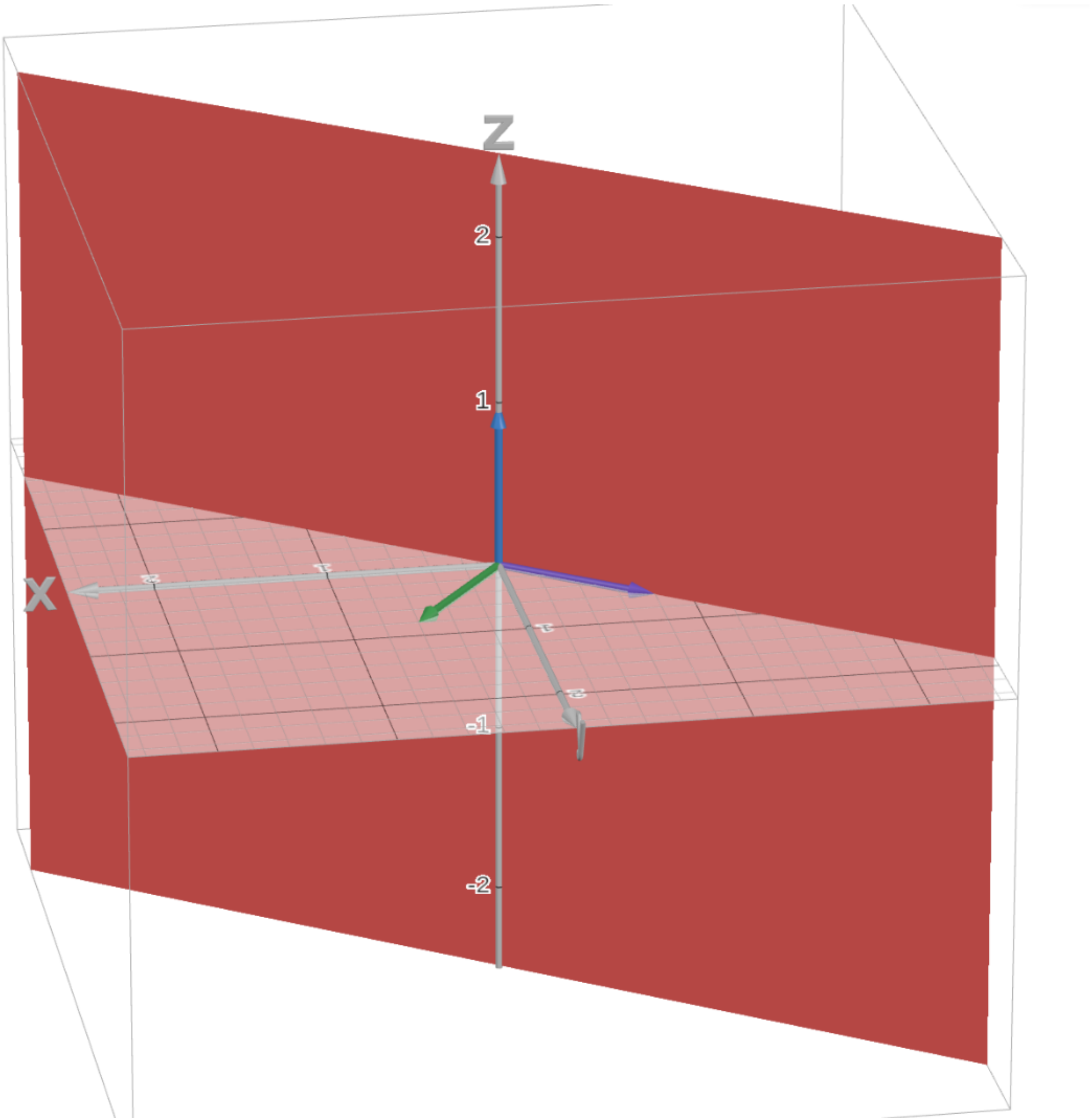
Such a matrix is called "householder matrix" and can be used to compute the QR decomposition (you will see this in NumCS!)

Alternatively, we can see this as a change of basis problem:
We seek to find a basis B where reflecting along P is a very simple operation – changing the sign of one component of $[x]_B$.

To achieve this, we want one basis vector to be a normal vector of P and the other two to span the plane P.

$$\text{A suitable basis is } B = \left\{ \underbrace{\begin{bmatrix}3/5\\4/5\\0\end{bmatrix}}_{\text{spans } P^\perp}, \underbrace{\begin{bmatrix}-4/5\\3/5\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}}_{\text{span } P} \right\}$$

(we can find it by e.g. rotating the first vector appropriately or sketching P)



We compute the following change of basis matrices

from the standard basis to B:

$$T = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

from B to the standard basis:

$$T^{-1} = \begin{bmatrix} -3/5 & -4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the reflection matrix in regard to the basis B:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Piecing this together results in:

$$B = T\,A\,T^{-1}$$

takes vector back to standard basis

reflects vector in basis B

takes vector in standard basis to B

References:

Last years course
Old exams for some of the quiz questions
Sergey Treil, Linear Algebra Done Wrong, https://www.math.brown.edu/streil/papers/LADW/LADW_2021_01-11.pdf