Linear Algebra
Week 14

Felix Breuer
G-04

Quiz

1. Let $a_1, \dots a_n$ be the columns of $A \in \mathbb{R}^{m\times n}$ and $b_1, \dots, b_n$ be the rows of $B \in \mathbb{R}^{n\times p}$. Then $AB = \sum_{i=1}^{n} a_i b_i$ TRUE
2. A is not invertible $\iff$ A has only eigenvalue 0 FALSE $\Rightarrow$ doesn't hold, it suffices that one eigenvalue is 0 (and others nonzero)
3. We can find the representation matrix A of a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ by taking $T(e_i)$ for all standard basis vectors e_i of R^n as the columns of A in any order. FALSE the order matters!
4. A and $S^{-1}AS$ have the same eigenvalues. TRUE can be proven with/without determinants! try for yourself
5. Per the spectral theorem any symmetric matrix is diagonalizable with positive real eigenvalues. FALSE A might have eigenvalues that are not positive
6. What does the spectral theorem state about a symmetric matrix $A \in \mathbb{R}^{n\times n}$? · A is diagonalizable with real eigenvalues, there exists an orthonormal basis of eigenvectors of $\mathbb{R}^n$, A has spectral decomp. $A = U \Lambda U^T$ with U orthogonal
7. A matrix is invertible if and only if it is diagonalizable. FALSE Neither direction holds. Consider counterexamples $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ for $\Rightarrow$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for $\Leftarrow$
8. Give an example of a matrix that is not diagonalizable over $\mathbb{C}$. A good example is often $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
9. Let $A = LDL^T$ where L is lower triangular and D diagonal with only positive entries along the diagonal. Prove that then A is positive definite.

   Hint: If $D = \text{diag}(d_1, \dots d_n)$ and all diagonal entries are positive, $D = D^{1/2}D^{1/2}$ where $D^{1/2} = \text{diag}(\sqrt{d_1}, \dots \sqrt{d_n})$

$$x^T A x = x^T L D L^T x = x^T L D^{1/2} D^{1/2} L^T x = x^T L (D^{1/2})^T D^{1/2} L^T x$$
$$= (D^{1/2} L^T x)^T (D^{1/2} L^T x) = \| D^{1/2} L^T x \|_2^2 \geq 0$$

and $\| D^{1/2} L^T x\|_2^2 = 0 \iff D^{1/2} L^T x = 0 \iff x = 0$ as $D^{1/2} L^T$ is invertible ($D^{1/2}$ is diagonal with nonzero diagonal entries, L is upper triangular with 1's on the diagonal).

Hence $x^T A x = \| D^{1/2} L^T x \|_2 > 0$ for all $x \neq 0$, A is PD.

We call a symmetric matrix $A \in \mathbb{R}^{n \times n}$

positive definite (PD)

if all its eigenvalues are positive

($\Longleftrightarrow x^T A x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$)

PD / PSD is also commonly defined like this

positive semidefinite (PSD)

if all its eigenvalues are nonnegative

($\Longleftrightarrow x^T A x \geq 0$ for all $x \in \mathbb{R}^n$)

Some key facts for SVD:

For any $A \in \mathbb{R}^{n \times n}$, $A^T A$ and $A A^T$

- are symmetric and positive semidefinite
- have the same nonzero (real!) eigenvalues

spectral theorem

The **singular values** of A are the square roots of eigenvalues of $AA^T / A^T A$.

We arrange them in $\Sigma \in \mathbb{R}^{m \times n}$ where $\Sigma_{ii} = \sigma_i$ is the ith largest singular value of A and all other entries of Σ are zero.

There are eigenvalue decompositions as follows:

$$AA^T = U \Sigma \Sigma^T U^T, \qquad AA^T u_i = \sigma_i^2 u_i$$

$$AA^T = V \Sigma^T \Sigma V^T, \qquad A^T A v_i = \sigma_i^2 v_i$$

where u_i, v_i are columns of U/V

U and V are orthogonal per the spectral theorem.

The singular value decomposition

Any $A \in \mathbb{R}^{m \times n}$ can be decomposed as

$$A = \underset{m\times m}{U} \; \underset{m \times n}{\Sigma} \; \underset{n \times n}{V}^T$$

$$U = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{bmatrix}$$

- columns $u_1, \dots, u_m$ are normalized eigenvectors of AA^T (left singular vectors)
- orthogonal, hence orthonormal columns and rows

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & \ddots & \\ 0 & & & & \underset{=\min\{m,n\}}{\sigma_p} \end{bmatrix}$$

- diagonal entries are singular values of A (square roots of eigenvalues of AA^T / A^TA)

$$\sigma_1 \geq \dots \geq \underset{=\text{rank}(A) = \#\text{ nonzero singular values}}{\sigma_r} > \sigma_{r+1} = \dots = \sigma_{\min\{m,n\}} = 0$$

$$V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

- columns $v_1, \dots, v_n$ (rows of V^T) are normalized eigenvectors of A^TA (right singular vectors)
- orthogonal, hence orthonormal columns and rows

Computing SVD:

If we have already computed U (V) we can get V (U) as follows:

$$A = U\Sigma V^T \iff AV = U\Sigma \iff Av_i = \sigma_i u_i \quad \text{for } i = 1, \dots, n$$

$$A = U\Sigma V^T \iff U^T A = \Sigma V^T \iff u_i^T A = \sigma_i v_i^T \quad \text{for } i = 1, \dots, m$$

Exercise Find the SVD of $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

We compute A^TA and AA^T along with their eigenvalues/eigenvectors:

$$AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad A^TA = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

AA^T and A^TA have nonzero eigenvalues $2, 1$

$\Rightarrow \sigma_1 = \sqrt{2}, \sigma_2 = \sqrt{1} = 1$ (can be read off from AA^T)

eigenvectors of AA^T: $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

For those of A^TA we use u_1, u_2:

$u_1^T A = \sigma_1 v_1^T, \quad u_2^T A = \sigma_2 v_2^T$

$\Rightarrow [1 \;\; 0 \;\; -1] = \sqrt{2}\, v_1^T$ which gives us $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}}$

$[0 \;\; 1 \;\; 0] = v_2^T$

$v_3 \in N(A), \; v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}}$

$AA^T u_1 = 2u_1, \; AA^T u_2 = u_2$

$A^TA v_1 = 2v_1, \; A^TA v_2 = v_2, \; A^TA v_3 = 0 v_3$

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = U^T, \quad V = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

Fundamental subspaces with SVD

$\text{span}\{u_1, \dots, u_r\} = C(A)$

$\text{span}\{u_{r+1}, \dots, u_m\} = N(A^T)$

$\text{span}\{v_1, \dots, v_r\} = C(A^T)$

$\text{span}\{v_{r+1}, \dots, v_n\} = N(A)$

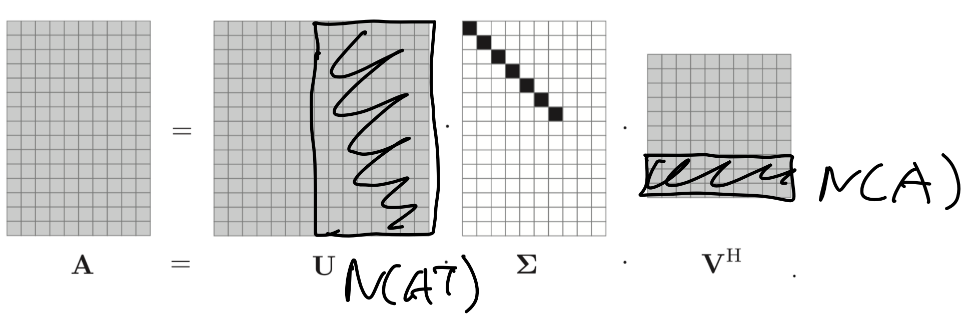
$AV = U\Sigma$

$A^T U = V \Sigma^T$

these equalities can be used to proof the statements on the left

Exercise HS 19

a) Die Singulärwertzerlegung der Matrix $\mathbf{A} \in \mathbb{C}^{15\times10}$ kann graphisch folgendermassen dargestellt werden: /6



Dabei entsprechen die grauen Kästchen Zahlen aus $\mathbb{C}$, die schwarzen stehen für reelle Zahlen > 0 und die weissen entsprechen der Zahl 0. An einer Singulärwertzerlegung dieser Form, lässt sich eine Basis des Kerns sowohl von $\mathbf{A}$ als auch von $\mathbf{A}^H$ ablesen. Markieren Sie die Kästchen, in denen sich die entsprechenden Vektoren befinden. Bitte machen Sie deutlich welche Markierung zu welchem Kern gehört.

The Pseudoinverse with SVD

$A^+ = V_r \Sigma_r^{-1} U_r^T$

Reduced SVD

As all singular values $\sigma_{r+1}, \dots, \sigma_p$ are zero, columns $r+1, \dots, p$ of U and V don't contribute to A, we can write

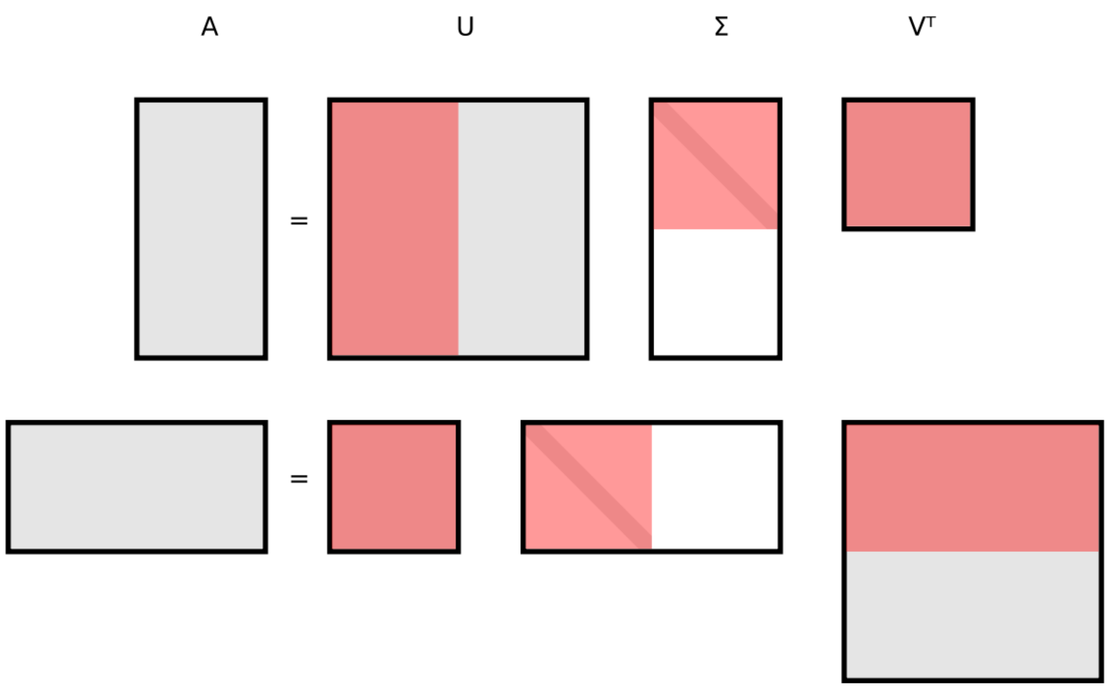
$$A = U \Sigma V^T = \underset{m \times r}{U_r} \; \underset{r \times r}{\Sigma_r} \; \underset{r \times n}{V_r}^T$$

first r columns of U

first r columns of Σ

first r rows of V^T

Examples



Application of SVD: Image Compression

Eckart - Young - Mirsky Theorem $\rightarrow$ best low rank approximation

A_k is best rank k approximation of A in terms of spectral norm

setting $\sigma_{k+1}, \dots, \sigma_p$ to 0

http://timbaumann.info/svd-image-compression-demo/

References:
Last years course
https://github.com/mitmath/1806
https://courses.grainger.illinois.edu/cs357/sp2021/notes/ref-16-svd.html