Linear Algebra Week 3

Felix Breuer

5/10/2023

1. Recap: True / False

1.1 Let $u, v, w \in \mathbb{R}^n$, $w \neq 0$ such that u and v are orthogonal to w. Then u and v are linearly dependent.

False: A counterexample is $u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
(in $\mathbb{R}^2$, it is true. You could try proving the case $w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$)

1.2 Any sequence that contains the zero vector is linearly dependent.

True: There is a linear combination of vectors in the sequence that equals the zero vector where not all scalars are 0: Let the sequence be given by $v_1, \dots, v_m$ with $v_k = 0$ for some $k \in \{1, \dots, m\}$.
$0 = 0 \cdot v_1 + \dots + \alpha v_k + \dots + 0 v_m = \alpha v_k = \alpha 0$ for any $\alpha \neq 0$, $\alpha \in \mathbb{R}$. Hence per definition, $v_1, \dots, v_m$ is linearly dependent.

1.3 Let $v_1, \dots, v_m$ be a sequence of vectors. If a subsequence of $v_1, \dots, v_m$ is linearly dependent, then so is $v_1, \dots, v_m$.

True: We have that some subsequence $v_{i_1}, \dots, v_{i_l}$ is linearly dependent. Per definition there are $\alpha_{i_1}, \dots, \alpha_{i_l}$:
$0 = \alpha_{i_1} v_{i_1} + \dots + \alpha_{i_l} v_{i_l}$. We add to this any vector in $v_1, \dots, v_m$ but not in $v_{i_1}, \dots, v_{i_l}$, multiplied with 0 and can still express 0 as a linear combination of $v_1, \dots, v_m$ where not all scalars are 0.

1.4 We can always make a set of linearly independent vectors dependent by adding a vector of our choice.

True: Adding the zero vector has this effect.

1.5 A hyperplane in $\mathbb{R}^4$ has dimension 3.
(will be defined more rigorously soon)

For $d \neq 0$: $\{ v \in \mathbb{R}^n \mid v \cdot d = 0 \}$

True: Hyperplane always has one dimension less than ambient space.

Note that with this definition, a hyperplane always passes through the origin and contains the zero vector.

2. Matrix-Vector and Matrix-Matrix Multiplication

2.1 Matrix-Vector Mul., column space, rank

We previously looked at how we can multiply a matrix $A \in \mathbb{R}^{m\times n}$ with a vector $x \in \mathbb{R}^n$ to get a vector $b \in \mathbb{R}^m$

Example: $\underset{\in \mathbb{R}^{2\times 3}}{A} = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 2 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \in \mathbb{R}^{3\times 1}$ we can see a column vector as a 3x1 matrix, row vector as 1x3 matrix

$$Ax = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3\cdot 1 + 2\cdot 2 + 5\cdot 0 \\ 1\cdot 1 + 2\cdot 2 + 4\cdot 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$= 1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2\cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} + 0\cdot \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$ linear combination of columns of A

$$= \begin{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \text{row } 1 \cdot x \\ \text{row } 2 \cdot x \end{bmatrix}$$ dot product of rows of A with x

Definition The column space of A, denoted $C(A)$, is the span (set of all linear combinations) of columns of $A \in \mathbb{R}^{m\times n}$

$$\boxed{C(A) = \{Ax \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m}$$

Definition The rank of A is the ~~number of linearly~~ independent columns of A (i.e. max cardinality of linearly independent subsequence of columns of A)

$$\boxed{\text{rank}(A) = \#\text{ independent columns}}$$

Remark: row space of A $R(A)$ is span of rows of A: $R(A) = \{xA \mid x \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$

2.2 | Special types of matrices, transpose
(some)

Identity Matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$a_{i,j} = 1 \Leftrightarrow i = j$
0 sonst

$IA = AI = A$

Zero matrix

$$0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$a_{i,j} = 0$ für alle i, j.

$0A = A0 = 0$

diagonal matrix

$$\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$a_{i,j} = 0$ für $i \neq j$

upper triangular matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$a_{i,j} = 0$ für $i > j$

lower triangular matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$a_{i,j} = 0$ für $i < j$

symmetric matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$a_{i,j} = a_{j,i}$

$(A = A^T)$ see below

"A gespiegelt an Diagonale = A"

Transpose

Definition: for all $i,j \in \{1, .., n\}$

$$\boxed{(A^T)_{i,j} = A_{j,i}}$$

-> erste Reihe wird zu erster Spalte...
-> Spalten von A^T zeilen von A

- Damit können wir Platz sparen: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [1 \ 2 \ 3]^T$
- Dot product: $x \cdot y = \sum_{i=1}^{n} x_i y_i = x^T y$
- $C(A) = R(A^T)$

2.3 Matrix-Matrix Multiplication

$$\underset{\in \mathbb{R}^{m\times n}}{A}\ \underset{\in \mathbb{R}^{n\times p}}{B} = (AB) \in \mathbb{R}^{m\times p}$$

only defined if these match!

$$(AB)_{i,j} = (\text{row } i \text{ of } A) \underset{\text{dot product}}{\cdot} (\text{column } j \text{ of } B)$$

$$(AB)_{i,j} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

- If we already know matrix vector multiplication, we already can compute $AB \Rightarrow$ generalization/extension

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} | & & | \\ b_1 & \cdots & b_p \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ Ab_1 & \cdots & Ab_p \\ | & & | \end{bmatrix}$$

Properties

- Associativity: $A(BC) = (AB)C$
- Distributivity: $A(B+C) = AB + AC$
- But **NOT** generally (usually) commutative! $AB \underset{\text{usually}}{\neq} BA$

Example

$$\begin{bmatrix} 3 & 4 & 1 \\ 3 & 4 & 2 \\ 4 & 4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 3 \\ 0 & 2 & 4 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \overset{= 3\cdot 3 + 4\cdot 0 + 1\cdot 3}{12} & 15 & 27 \\ 15 & 16 & 29 \\ 18 & 18 & 32 \end{bmatrix}$$

- column j of AB is ~~are~~ linear combination of columns of A with ~~column~~ scalars from column j of B:

$$3\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 15 \\ 18 \end{bmatrix} \longrightarrow \text{first column of example product}$$

- row i of AB is linear combination of rows of B with respective scalars from row i of A:

$$3\,[3\ 2\ 3] + 4\,[0\ 2\ 4] + 1\cdot[3\ 1\ 2] = [12\ \ 15\ \ 27]$$

$\downarrow$ first row of example product

3. The CR decomposition

$$\boxed{A = C \quad R}$$

C: first r independent columns

R: contains information on how to combine columns C to get A

How to compute it (we will see faster way later)

- Go through columns of A:
  - ↳ add column A if it is not linear combination of previous ones to C
  - ↳ for each column, add information to R on how it is attained from combining columns of C

(Example)

$$A = \begin{bmatrix} 3 & -3 & 1 & 8 & 0 \\ 2 & -2 & 0 & 4 & 0 \\ 4 & -4 & 0 & 8 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

col 4 of A is $2 \times$ col 1 + $2 \times$ col 2 of C

col 2 of A is −1 col 1 of C

→ A is attained from:
- combinations of columns of C (see 2.3)
- combinations of rows of R

- columns of C span $C(A)$
- rows of R span $R(A) = C(A^T)$

Shows us that # independent cols of A = # independent rows of A
= rank (A)

- r columns of C independent by construction
  → columns of A are combination of cols of C
- r rows of R independent (identity matrix contained in R)
  → every row of A combination of rows of R

Proof Claim: The choice of R in CR decomp. is unique

Assume there are $\underbrace{A, C, R, R'}_{\text{according to def of CR decomp.}}$ such that $A = CR = CR'$

$\Rightarrow CR - CR' = 0$ (Matrix Addition)

$\Rightarrow C(R - R') = 0$ (Distributivity)

$\Rightarrow Cw = 0$ for every column vector w of $(R - R')$ (def zero matrix)

$\Rightarrow$ As C ~~is linearly~~'s columns are linearly independent, only x such that $Cx = 0$ is $x = 0$

$\Rightarrow w = 0$ for any column w of $(R - R') \Rightarrow (R - R') = 0$

$\Rightarrow R = R'$. Hence the choice of R is unique.

4. Gaussian Elimination

$n=3$ unknowns

$m=3$ equations

$$\begin{aligned} x + y + 3z &= 3 \\ 2x + y + 3z &= 1 \\ 3x + 3y + 2z &= 2 \end{aligned}$$

↓ Matrix-Vector Multiplication

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

↓ variable names don't matter if each variable has its own position (column)

"extended matrix"
$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 2 & 1 & 3 & 1 \\ 3 & 3 & 2 & 2 \end{array}\right]$$

definition follows!

Idea Reduce to row echelon form ("Zeilenstufenform"), then we can easily compute solution with back substitution

Allowed ops

1) swap rows
2) Multiply row with scalar
3) Add scalar multiple of one row to another

All of these are sound: They are reversible, solution set doesn't change

⟶ They correspond to multiplying extended matrix from left with elementary matrices

Getting these matrices is easy! We apply the operation on the identity matrix:

e.g. swap rows 1 and 2:
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

multiply row 2 by 42:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 42 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

subtract 3 times row 2 from row 3
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = E_{32}$$

$$\begin{array}{c} \\ -2 \\ -3 \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 2 & 1 & 3 & 1 \\ 3 & 3 & 2 & 2 \end{array}\right]$$

Mark 1 as pivot*. We now want to eliminate 2 and 3 below 1. We do this by subtracting the corresponding amount of row 1.

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & -1 & -3 & -5 \\ 0 & 0 & -7 & -7 \end{array}\right]$$

We are lucky! Only one elimination step was needed for REF. Now we can <u>backsubstitute</u>.

* pivot = first nonzero entry of row

$$-7z = -7 \Rightarrow z = 1$$
$$-y - 3z = -5 \Leftrightarrow -y = -2 \Leftrightarrow y = 2$$
$$x + y + 3z = 3 \Leftrightarrow x + 2 + 3 = 3 \Leftrightarrow x = -2$$

Hence our solution vector is

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

<u>row echelon form</u> (REF)

1. All zero rows at bottom
2. First nonzero entry of row is strictly to the right to first nonzero element of row above

(This implies that below a pivot all entries are 0)

<u>reduced if also</u>

3. each pivot is 1
4. all entries aside pivot in each column are 0

We call this reduced row echelon form (RREF) then.

$$\begin{bmatrix} 1 & 0 & x & 0 \\ 0 & 1 & x & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

example of reduced row echelon form

Exercise Solve the following using Gaussian Elimination:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{array}\right]$$

(solution: $x = [1, 1, -1]^T$)

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 2 & -3 & 1 & 6 \\ 3 & -5 & 0 & 7 \\ 1 & 0 & 5 & 9 \end{array}\right]$$

(solution: $x = \begin{bmatrix} 4 + 7x_3 \\ 3x_3 + 4 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}$

References:
Gilbert Strang, Introduction to Linear Algebra, 6th Edition, Wellesley - Cambridge Press.
Sergey Treil, Linear Algebra Done Wrong, https://www.math.brown.edu/streil/papers/LADW/LADW_2021_01-11.pdf