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Linear Algebra
Week 4

TRUE / FALSE

1. The dot product of two vectors $v \cdot w$ (in $\mathbb{R}^n$) can be computed as vw^T

FALSE

$v \cdot w = v^T w$, not vw^T

$vw^T \in \mathbb{R}^{n \times n}$ has dimensions $n \times n$ and is called outer product. This matrix always has rank 1 (recall the column/row picture: the columns of AB are linear combinations of the columns of A, the rows of AB combinations of the rows of B) and some other interesting properties we might see later (at most one nonzero eigenvalue).

$$\underset{\overset{\shortparallel}{A}}{\begin{bmatrix} — a_1 — \\ \vdots \\ — a_m — \end{bmatrix}} \underset{\overset{\shortparallel}{B}}{\begin{bmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{bmatrix}} = \begin{bmatrix} | & & | \\ Ab_1 & \cdots & Ab_n \\ | & & | \end{bmatrix} = \begin{bmatrix} — a_1 B — \\ \vdots \\ — a_m B — \end{bmatrix}$$

2. If vectors $u, v, w \in \mathbb{R}^n$ are pairwise linearly independent, u, v, w is linearly independent as well

FALSE pairwise linear independence $\not\Rightarrow$ linear independence

Counterexample:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3$$

3. There is no matrix $A \in \mathbb{R}^{2\times 3}$ with $\text{rank}(A) = 3$

TRUE

$$\text{rank}(A) \leq \min\{m, n\},\ A \in \mathbb{R}^{m\times n}$$

Different ways to see this:

- $\text{rank}(A) = \text{rank}(A^T)$ → see week 3 notes, the CR decomposition helps us see this
- columns are vectors in $\mathbb{R}^2$, max. cardinality of sequence of vectors from $\mathbb{R}^2$ is 2

Gauss-Elimination: Infinite number of solutions

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

← This (extended) matrix is already in reduced row echelon form (RREF). Hence we apply back-substitution:

variables corresponding to these columns are free

definition

row echelon form (REF)

1. All zero rows at bottom
2. First nonzero entry of row is strictly to the right to first nonzero element of row above

(This implies that below a pivot all entries are 0)

reduced if also

3. each pivot is 1
4. all entries aside pivot in each column are 0

We call this reduced row echelon form (RREF) then.

We get:
$$\begin{cases} x_1 = 1 - 2x_2 \\ x_2 \text{ is free (no pivot)} \\ x_3 = 2 - 5x_4 \\ x_4 \text{ is free} \\ x_5 = 3 \end{cases}$$

This gives us the following solution vector x:

$$x = \begin{bmatrix} 1 - 2x_2 \\ x_2 \\ 2 - 5x_4 \\ x_4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix},\quad x_2, x_4 \in \mathbb{R}$$

We seperate occurrances of free variables into separate vectors.

Solution set of a system of linear equations (SLE)

We consider a SLE given by $Ax = b$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ with the solution vector $x \in \mathbb{R}^n$.

- no solution

  if there is a row $[0 \; \dots \; 0 \mid x]$, $x \neq 0$ in the extended matrix $\longrightarrow$ a contradiction "consistency condition violated"

- exactly one solution

  if $\operatorname{rank}(A) = n$ (# columns) and consistency condition (above) fulfilled
  ($\Longleftrightarrow$ A is invertible, see next page)

- else: infinitely many solutions

  with $n - \operatorname{rank}(A)$ free variables

We call $Ax = 0$ the homogenous system of equations
$\longrightarrow$ always has solution $x = 0$

The inverse of a matrix A

Definition We call $A \in \mathbb{R}^{n \times n}$ **invertible** if there exists $A^{-1} \in \mathbb{R}^{n\times n}$ such that $AA^{-1} = A^{-1}A = I$

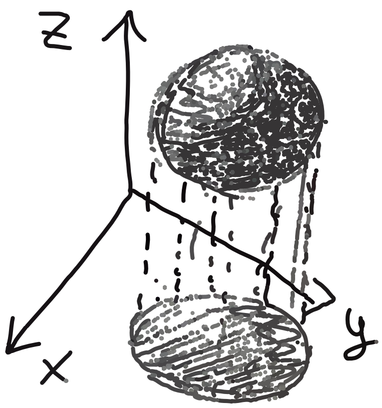
In the real numbers ($\sim \mathbb{R}^{1\times 1}$) the inverse of some $a \neq 0$ is $\frac{1}{a}$.
Considering matrices as functions $x \mapsto Ax$ a matrix is invertible $\Leftrightarrow$ its corresponding linear map (we will define this soon) i.e. effect on a vector x is reversible.

The following is an example of a non-invertible matrix:

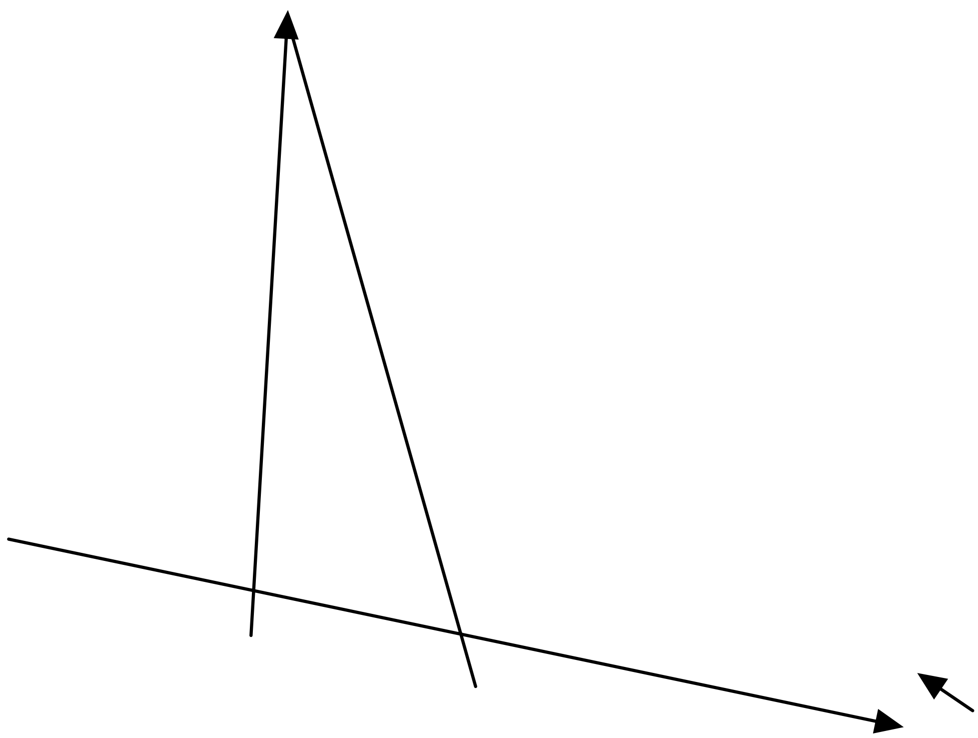
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A projects vectors in $\mathbb{R}^3$ onto the xy-plane

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$



This linear map is not reversible, we lose the depth (z-coordinate) of any input vector.



Extension of Inverse Theorem $\quad A \in \mathbb{R}^{n\times n}$

from lecture $\begin{cases} 1. & A \text{ is invertible} \\ 2. & \text{For all } b \in \mathbb{R}^n \; Ax=b \text{ has unique solution } x\in\mathbb{R}^n \\ 3. & \text{The columns of } A \text{ are independent} \end{cases}$

4. $Ax = 0 \Rightarrow x = 0$
5. $RREF(A) = I$
6. $A = \underbrace{(E_k \cdots E_1)^{-1}}_{\text{product of elementary matrices (from elimination)}} I$

<u>2) ⇒ 1)</u> $\overset{(i)}{AB} = I$ was already proven in the lecture.
Further we can assume $3_{(ii)}$ (2) ⇒ 3) already proven)

<u>Claim</u>: It follows $BA = I$

$A = IA \overset{(i)}{=} (AB)A \overset{\text{distr.}}{=} A(BA)$. Hence $A - A(BA) = 0$.

$\Rightarrow A(I - BA) = 0$ (distr. law)

$\Rightarrow Aw = 0$ for any column w of $(I - BA)$

As the columns of A are independent (ii) $Ax = 0 \Rightarrow x = 0$.

$\Rightarrow w = 0$ for all columns w of $(I - BA)$

$\Rightarrow I - BA = 0 \Rightarrow I = BA$

<u>3 ⇒ 4</u> This follows from the definition of linear independence and matrix-vector multiplication.

<u>4 ⇒ 5</u> Rank $(A) = n$, there are no free variables (columns without pivot)
⇒ every row and column has a pivot
⇒ $RREF(A) = I$ per definition RREF

<u>5 ⇒ 6</u> We get this through applying the Gauss-Algorithm - read ahead!

Finding Inverses

Idea:

$$A = IA$$
$$E_1 A = E_1 I A$$
$$\vdots$$
$$I = \underbrace{E_k \cdots E_1 I}_{= A^{-1}} A$$

Note:
E_i = ith elimination matrix in Gauss Elimination

We generally write E_{ij} for the elimination matrix that adds multiples of row j to row i

We apply elimination matrices from the left until we get $RREF(A) = I$ (If A is not invertible we will not get I). The product of these elimination matrices is A^{-1}.

$$[A \mid I] \overset{\text{Gauss Elimination}}{\rightsquigarrow} [I \mid A^{-1}]$$

1. Write down A next to the identity matrix
2. Apply elimination until the left side is the identity matrix
3. The matrix on the right side is now A^{-1}

If A is not invertible the above fails: We cannot get the identity matrix, there will be a row of zeroes on the left side.

Example: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 0 \\ 3 & 4 & 1 \end{bmatrix}$

The circled numbers are pivot elements, the values next to the matrix signify how many times we add the pivot row to the row the number is next to.

$$\begin{array}{c} \\ -3 \\ -3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & 5 & 0 & 0 & 1 & 0 \\ 3 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} 2 \\ \\ -2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -3 & 1 & 0 \\ 0 & -2 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 2 & 0 \\ 0 & -1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} -5 & 2 & 0 \\ 3 & -1 & 0 \\ 3 & -2 & 1 \end{matrix}}_{= A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

elimination matrices

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \to \underset{E_{31}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}} \to$$

$$\underset{E_{12}}{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \to \underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \to$$

$$\underset{E_{22}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Keep in mind elimination matrices don't commute in general! The multiplication order matters.

$$I = \underbrace{E_{22} E_{32} E_{12} E_{31} E_{21}}_{= A^{-1}} A$$

LU decomposition

← sehr relevant!

random fact: super computer rankings are based on it

$$A = LU$$

lower triangular (product of inverses of elimination matrices)

upper triangular (REF from elimination)

Applying Gauss-Elimination gives us:

$$E_k \cdots E_1 A = U$$

$$\Longleftrightarrow \quad A = (E_k \cdots E_1)^{-1} U$$

$$= E_1^{-1} \cdots E_k^{-1} U$$

$$= LU$$

For now we restrict ourselves to only adding multiples of one row to another (no row swaps / scaling):

row swaps: can be added by keeping track of row permutations

scaling: not needed for REF

Example: Find the LU decomposition of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

Whenever we eliminated an entry we put the amount we subtracted from that row in a box in place of the zero.

$$\begin{array}{r} \\ -1 \\ -2 \end{array}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{array}{r} \\ \\ 3 \end{array}\begin{bmatrix} 1 & 2 & 3 \\ \boxed{1} & 1 & -1 \\ \boxed{2} & -3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ \boxed{1} & 1 & -1 \\ \boxed{2} & \boxed{-3} & -8 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{bmatrix}$$ is the upper triangular matrix we get from elimination

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$ is obtained from taking the boxed entries and adding 1's across the diagonal

Writing the matrices out explicitly gives us:

$$E_{32}E_{31}E_{21}A = U$$
$$\Longleftrightarrow \quad A = (E_{32}E_{31}E_{21})^{-1}U$$
$$= E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$$
$$= LU$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{bmatrix}$$

$$\Longleftrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{bmatrix}$$

References

Previous iteration of the course for some of the examples

Sergey Treil, Linear Algebra Done Wrong, https://www.math.brown.edu/streil/papers/LADW/LADW_2021_01-11.pdf (example on page 2)

Also shoutout to Sergey Prokudin, my TA from last year, for his great exercise sessions