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G-04

Linear Algebra
Week 5

True | False

1. Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$. If AB is invertible, then A and B are invertible.

FALSE

Counterexample:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ is invertible but neither A nor B is.

However: If $A, B \in \mathbb{R}^{n \times n}$, the statement is **TRUE**
($\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$)
Again, the row/column picture is helpful here to see why this holds:

$$\underbrace{\begin{bmatrix} - & a_1 & - \\ & \vdots & \\ - & a_m & - \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{bmatrix}}_{B} = \begin{bmatrix} | & & | \\ Ab_1 & \cdots & Ab_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1 B & - \\ & \vdots & \\ - & a_m B & - \end{bmatrix}$$

2. $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ **TRUE**

Inverse operation of adding multiple of one row to another is subtracting that multiple again

LU decomposition (continued)

$$\begin{aligned} & E_k \cdots E_1 A = U \\ \iff \quad & A = (E_k \cdots E_1)^{-1} U \\ & \quad = E_1^{-1} \cdots E_k^{-1} U \\ & \quad = LU \end{aligned}$$

• Row permutations

In some cases it's necessary to swap rows when performin Gauss-Elimination (zeroes in pivot position!).
In these cases we keep track of all row exchanges in a separate matrix P:

$$PA = LU$$

0 pivot!

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=P} \underbrace{\begin{bmatrix} 0 & 2 & 2 \\ 2 & 4 & 3 \\ 1 & -1 & 2 \end{bmatrix}}_{=A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & -3/2 & 1 \end{bmatrix}}_{=L} \underbrace{\begin{bmatrix} 2 & 4 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 7/2 \end{bmatrix}}_{=U}$$

• Solving $Ax=b$

$$Ax=b \implies PAx = Pb \implies PL\underbrace{Ux}_{=c} = Pb$$

1. Solve $Lc = Pb$ for c
2. Solve $Ux = c$ for x

Example

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

position of rows

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 3 & 4 & 3 \end{array}\right] \sim \begin{array}{c} \\ \\ -2 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 2 & 3 & 4 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ \boxed{0} & 1 & 1 & 1 \\ \boxed{2} & \boxed{3} & -1 & 3 \end{array}\right]$$

-3

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

first and second row swapped

solve for $b = \begin{bmatrix} 2 \\ 3 \\ 11 \end{bmatrix}$, $\quad Pb = \begin{bmatrix} 3 \\ 2 \\ 11 \end{bmatrix}$

1. $Lc = Pb$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 11 \end{bmatrix}$$

$c_1 = 3$
$c_2 = 2$
$2c_1 + 3c_2 + c_3 = 11$
$\iff 6 + 6 + c_3 = 11$
$\iff c_3 = -1$

2. $Ux = c$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$x_3 = 1$
$x_2 + 1 = 2 \implies x_2 = 1$
$x_1 + 1 = 3 \implies x_1 = 2$

$$x = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Why LU decomposition is useful:

We only have to perform elimination once $(O(n^3))$ and can then solve $Ax=b$ for any vector b in $O(n^2)$ by forward/backward elimination.

Vector Spaces

So far we only considered vectors as elements of $\mathbb{R}^n$. We can define vectors in a more abstract way by describing the properties of vector addition and scalar multiplication:

vectors   scalars

A vector space is a set V over a field $\mathbb{F}$ along with two operations

$+ : V \times V \longrightarrow V$ (vector addition)
$\cdot : \mathbb{F} \times W \longrightarrow V$ (scalar multiplication)

linear combinations closed under these operations

such that the following 8 axioms are true:
Let $u, v, w \in V$, $\alpha, \beta \in \mathbb{F}$.

1. $v + w = w + v$
2. $(u+v)+w = u+(v+w)$
3. there is $0 \in V$: $v + 0 = 0$
4. there is $-v \in V$: $v + (-v) = 0$

vector addition

5. there is $1 \in V$: $1 \cdot v = v$
6. $\alpha(\beta v) = (\alpha\beta)v$

scalar multiplication

7. $\alpha(u+v) = \alpha u + \alpha v$
8. $(\alpha + \beta)u = \alpha u + \beta u$

both scalar multiplication, vector addition

We don't need to memorize these – they are the properties of vector addition and scalar multiplication already familiar to us.

Examples

- $\mathbb{C}$ over $\mathbb{R}$
- P_n, the set of polynomials of degree $\leq n$ $= \{a_n x^n + \dots + a_1 x + a_0 \mid a_n, \dots, a_0 \in \mathbb{R}\}$

  for some fixed n
- $\mathbb{R}^{n \times n}$, the set of $n \times n$ matrices

- We consider $V = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 \right\}$ with the operations:

$$+ : \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right) \mapsto \begin{bmatrix} a+c \\ b-d \end{bmatrix}$$

$$\cdot : \left(\alpha, \begin{bmatrix} a \\ b \end{bmatrix} \right) \mapsto \begin{bmatrix} \alpha a \\ b \end{bmatrix}$$

Is this a vector space? No! Not commutative.

$U \subseteq V$ is a **subspace** of V if it is a vector space

We can check this as follows:

1. U is non-empty: $0 \in U$
2. U is closed under vector addition
   For any $u, v \in U$: $u + v \in U$

   scalar multiplication
   For any $u \in U, \alpha \in F$: $\alpha u \in U$

this is how we usually prove U is a subspace

Examples

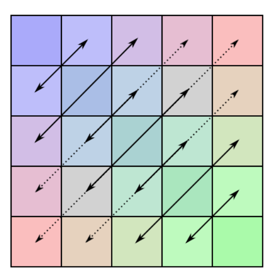
- $\{0\}, V$: **always** subspaces of a vector space V
- $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ is **not** a subspace of $\mathbb{R}^2$
  $\longrightarrow$ not closed under vector addition, scalar multiplication

Exercises

Find a subspace of $\mathbb{R}^{2x}$ that contains $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ but not $\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$ $\longrightarrow$ e.g. $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \in \mathbb{R}^2 \middle| a \in \mathbb{R} \right\}$

Remark:

If $A = A^T$ we call A **symmetric**



If $A = -A^T$ we call A **skew-symmetric**

Claim: $U = \{ A \in \mathbb{R}^{n\times n} \mid \overbrace{A = -A^T}^{\text{skew-symmetric}} \}$ is a subspace of $\mathbb{R}^{n\times n}$.

Proof:

We check the two properties 1. U is non-empty 2. U is closed under vector addition and scalar multiplication:

1. $0 = -0^T \Rightarrow 0 \in U$
2. Let $A, B \in U, \alpha \in \mathbb{R}$
   $\oplus$: $A + B = -A^T - B^T = -(A+B)^T \Rightarrow A+B \in V$
   $\odot$: $\alpha A = \alpha(-A^T) = \alpha((-1)A^T) = (\alpha(-1))A^T = -\alpha A^T \in U$

Hence U is a subspace of $\mathbb{R}^{n\times n}$.

$\mathcal{B} = \{v_1, \dots, v_k\} \subseteq V$ is a **basis** of V if:

- $\mathrm{span}(\{v_1, \dots, v_k\}) = V$
- $\{v_1, \dots, v_k\}$ is linearly independent

Every vector in V can be uniquely expressed as a linear combination of vectors from $\mathcal{B}$.

The **dimension** of V, denoted $\dim V$ is the cardinality (size) of any basis of V.

Examples There are usually many choices for a basis (infinitely many)

- $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is the standard basis of $\mathbb{R}^2$ but as you saw in the first exercise sheet $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is a basis as well!

- $\dim \mathbb{R}^n = n$
- $\dim P_n(x) = n+1$ with the standard basis $\{1, x, \dots, x^n\}$
- $\mathbb{C}$ over $\mathbb{R}$ forms a vector space of dimension 2 with basis $\{1, i\}$
- $\mathbb{C}$ over $\mathbb{C}$: dimension 1, basis is $\{1\}$
- $\dim \mathbb{R}^{n\times n} = n^2$

Exercise Find a basis of $U = \{A \in \mathbb{R}^{2\times 2} \mid A = -A^T\}$

Let $A \in U$. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -A^T = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix}$

This gives us: $a = -a \overset{+a}{\Longrightarrow} 2a = 0 \overset{:2}{\Longrightarrow} a = 0$

$d = -d \Longrightarrow d = 0$

$b = -c$

$\Longrightarrow A = \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} = c \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Hence a potential basis is $\mathcal{B} = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$.

General approach:

$\rightarrow$ To find basis of subspace, consider the definition of that subspace. # free variables = dim. We can form an equation with all free variables and then set one free variable to 1, all others to 0. Doing this for each free variable gives us a basis of the subspace:

Dimension of subspaces of (skew) symmetric matrices

- $\dim \{A \in \mathbb{R}^{n\times n} \mid A = -A^T\} = \frac{n^2 - n}{2} = \frac{n(n-1)}{2}$

n^2 entries, diagonal is $0 \Rightarrow n$ options less. lower triangle determines upper triangle $\rightarrow$ we can choose half the entries left

- $\dim \{A \in \mathbb{R}^{n\times n} \mid A = A^T\} = \frac{n^2 - n}{2} + n = \frac{n(n+1)}{2}$

we can choose the diagonal entries

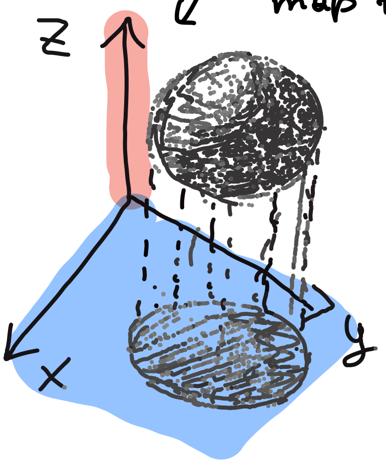
Two important subspaces: $C(A)$ and $N(A)$

Let $A \in \mathbb{R}^{m \times n}$: (or $A: V \to W$)

column space, image or range — $C(A) = \{ Ax \mid x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$

null space or kernel — $N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \} \subseteq \mathbb{R}^n$

vectors that map to 0 → $N(A)$



Claim: $C(A)$ is a subspace of $\mathbb{R}^m$

1. $0 \in \mathbb{R}^n \Rightarrow A \cdot 0 = 0 \in C(A)$
2. Let $u, v \in C(A)$
   $\Rightarrow Ax = u,\ Ay = v$ for $x, y \in \mathbb{R}^n$
   $u + v = Ax + Ay = A(x+y) \in C(A)$
   $\alpha u = \alpha Ax u = A(\alpha u) \in C(A)$

Hence the claim holds.

Claim: $N(A)$ is a subspace of $\mathbb{R}^n$

1. $A \cdot 0 = 0 \Rightarrow 0 \in N(A)$
2. Let $u, v \in N(A)$
   $\Rightarrow Au = Av = 0$
   $Au + Av = A(u+v) = 0$
   $\Rightarrow (u+v) \in N(A)$

Hence the claim holds.

References
Previous iteration of the course for some of the examples
Sergey Treil, Linear Algebra Done Wrong, https://www.math.brown.edu/streil/papers/LADW/LADW_2021_01-11.pdf (example on page 2)

Also shoutout to Sergey Prokudin, my TA from last year, for his great exercise sessions