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G-04

Linear Algebra
Week 6

TRUE / FALSE

1. Let V be a vector space. Then any basis of V has the same cardinality.

TRUE Proof: Let $\mathcal{B}_1, \mathcal{B}_2$ be two bases of V.
We can reduce any set of vectors that spans V to a basis (by computing a basis of the column space of $\mathcal{B}$ written as the columns of a matrix).

$\Rightarrow$ cardinality of basis $\leq$ cardinality of spanning set

$\Rightarrow |\mathcal{B}_1| \leq |\mathcal{B}_2|$ and $|\mathcal{B}_2| \leq |\mathcal{B}_1|$

$\Rightarrow |\mathcal{B}_1| = |\mathcal{B}_2|$

Hence the claim holds as we have shown that two arbitrary bases of V have the same cardinality.

2. The basis of a vector space is unique.

FALSE Counterexample: $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}7\\1\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}\right\}$ are both bases of $\mathbb{R}^2$

3. $A \in \mathbb{R}^{m \times n}$ has full (maximal) rank $\Leftrightarrow$ $\operatorname{rank}(A) = \min\{m, n\}$

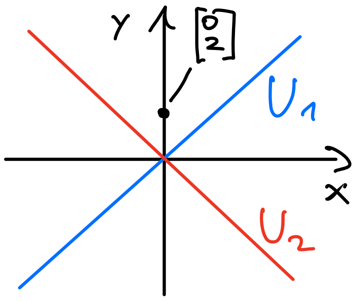
TRUE Intuitive reasoning: A matrix can't have more pivots (// rank) than the number of rows/columns

4. If U_1 and U_2 are subspaces of a vector space V, $U_1 \cup U_2$ is also a subspace of V.

FALSE

Counterexample: $V = \mathbb{R}^2$, $U_1 = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} \,\middle|\, \alpha \in \mathbb{R} \right\}$

$U_2 = \left\{ \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} \,\middle|\, \alpha \in \mathbb{R} \right\}$



$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in U_1$, $\begin{bmatrix} -1 \\ 1 \end{bmatrix} \in U_2$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \notin U_1 \cup U_2$, hence not a subspace.

Instead of considering the union of subspaces, it makes more sense to look at their sum: $U_1 + U_2 = \{ u_1 + u_2 \mid u_1 \in U_1 \text{ and } u_2 \in U_2 \}$ is always a subspace – in fact the smallest that contains both U_1 and U_2.

The following is often of special interest: If any $v \in V$ has a unique representation $v = u_1 + u_2$ with $u_1 \in U_1$, $u_2 \in U_2$, we call U_1 and U_2 complementary subspaces ($\Longleftrightarrow V = U_1 + U_2$ and $U_1 \cap U_2 = \{0\}$)
Then V is the direct sum $V = U_1 \oplus U_2$ of U_1 and U_2.

checking this is not enough if we consider more than 2 subspaces generally: $U_i \cap \sum_{j \neq i} U_j = \{0\}$

In the example from above $U_1 \oplus U_2 = \mathbb{R}^2$

$\mathbb{R}^3 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \oplus \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \oplus \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

5. Let $A \in \mathbb{R}^{n \times n}$, $\{b_1, \dots, b_n\} \subseteq \mathbb{R}^n$. There is an efficient algorithm that lets us solve $Ax = b_i$ for all $i \in \{1, \dots, n\}$ with total cost $O(n^3)$.

TRUE

Computing A's LU decomposition once in $O(n^3)$ and then solving $Lc = Pb$ and $Ux = c$ n times in $O(n^2)$ gives us total cost $O(n^3)$.

Two important subspaces: $C(A)$ and $N(A)$

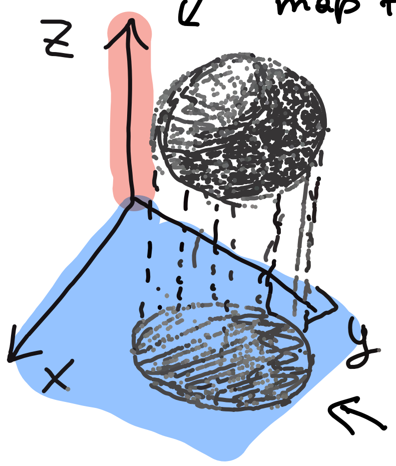
Let $A \in \mathbb{R}^{m \times n}$: (or $A: V \to W$) (continued!)

column space, image or range — $C(A) = \{ Ax \mid x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$

null space or Kernel — $N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \} \subseteq \mathbb{R}^n$

Examples

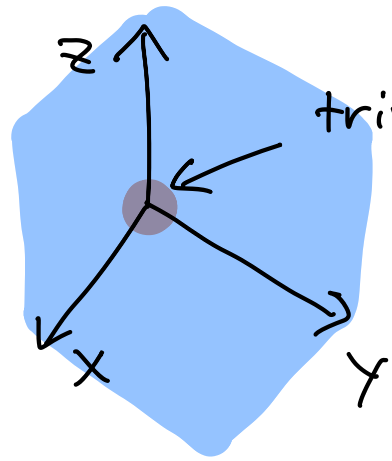
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$\dim C(A) = 2$

$\dim N(A) = 1$

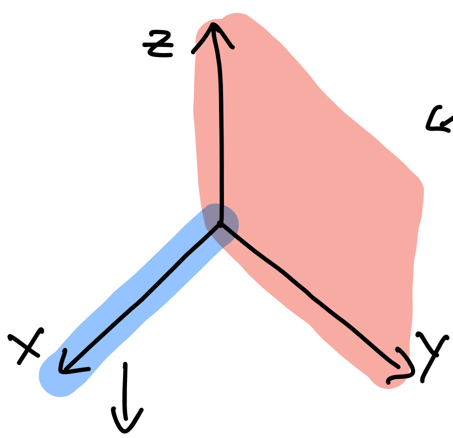
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$\dim C(A) = 3$

$\dim N(A) = 0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$N(A)$, all $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ with $x = 0$

$\dim C(A) = 1$

$\dim N(A) = 2$

Claim: $C(A)$ is a subspace of $\mathbb{R}^m$

1. $0 \in \mathbb{R}^n \Rightarrow A \cdot 0 = 0 \in C(A)$
2. Let $u, v \in C(A)$

$\Rightarrow Ax = u,\ Ay = v$ for $x, y \in \mathbb{R}^n$

$u + v = Ax + Ay = A(x+y) \in C(A)$

$\alpha u = \alpha A x u = A(\alpha u) \in C(A)$

Hence the claim holds.

Claim: $N(A)$ is a subspace of $\mathbb{R}^n$

1. $A \cdot 0 = 0 \Rightarrow 0 \in N(A)$
2. Let $u, v \in N(A)$

$\Rightarrow Au = Av = 0$

$Au + Av = A(u+v) = 0$

$\Rightarrow (u+v) \in N(A)$

Hence the claim holds.

Find basis of: ← this is a very typical exam exercise

– $C(A)$

- Apply Gaussian Elimination
- take the columns of A that have a pivot in REF (the original matrix)

– $N(A)$

- solve $Ax = 0$ (with Gaussian Elimination)
- $\dim N(A) = \#$ free variables
- For each free variable take the vector that it multiplies with to get $Ax = 0$ (set that free variable to 1, all others to 0, repeat)

Why is this sound?

Applying Gaussian Elimination on A to get $REF(A)$, i.e. multiplying with multiplication matrices from left

$N(A)$: doesn't affect solution set of underlying SLE

$\longrightarrow Ux = \tilde{b}$ has same solutions as $Ax = b$,

specifically for $b = 0$

↳ we have to apply elimination matrices on b as well!

$C(A)$: preserves linear dependence relations between colums we consider an arbitrary set B of columns of A.

Proof: This set is linearly dependent $\Longleftrightarrow$ there exists a nontrivial linear combination of them that is equal to zero, i.e. $Ax=0$ where x determines a linear combination of columns in B.

$REF(A) = EA = U$

We find $EAx = 0 \Longleftrightarrow Ax=0$ as E is invertible.

Hence any set B of columns of A is linearly independent if and only if the same columns in $ref(A)$ are linearly independent.

We also prove that the pivot columns span $C(A)$:

Claim: Columns of A that have pivot in $ref(A)$ span $C(A)$

Proof: Let $v_1, \dots, v_r$ be the columns with pivot in $REF(A) = EA = U$ and v be an arbitrary column of A.

$\Rightarrow Ev = E\alpha_1 v_1 + \dots + E\alpha_r v_r$

$\Rightarrow v = \alpha_1 v_1 + \dots + \alpha_r v_r$ (mul. with E^{-1})

Hence any column of A can be expressed as a linear combination of pivot columns.

$\Rightarrow span(v_1, \dots, v_r) = C(A)$

Exercises Find $C(A)$, $N(A)$ of the following matrices:

1) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$

$\text{ref}(A) = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

$C(A) = \text{span}\left(\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}\right)$

$x_1 + 2x_2 + 4x_3 = 0 \iff$

$x_1 = -2x_2 - 4x_3$

$x = \begin{bmatrix} -2x_2 - 4x_3 \\ x_2 \\ x_3 \end{bmatrix}$

$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ basis for $N(A)$ is

$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$

2) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix} = B$

$\text{ref}(B) = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix}$

A basis of $C(A)$ is

$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$

$x_2 = 0$

$x_1 + 2x_2 + 4x_3 = 0$

$= x_1 + 4x_3$

$\Rightarrow x = \begin{bmatrix} -4x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ basis for $N(A)$ is

$\left\{ \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$

Remark: A basis of a subspace U of a vector space V usually (unless $U = V$) has less elements than a basis of V. It's important that a basis only has elements that are actually in the subspace. Ensuring that the span of the basis vectors is exactly U is sufficient.

References

https://github.com/mitmath/1806 for one of the examples

Sergey Treil, Linear Algebra Done Wrong, https://www.math.brown.edu/streil/papers/LADW/LADW_2021_01-11.pdf

Sheldon Adler, Linear Algebra Done Right, https://link.springer.com/book/10.1007/978-3-319-11080-6