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G-04

Linear Algebra
Week 8

The exercise and quiz discussed can be found here: felixgbreuer.com/recap

The following pages contain solutions to the recap exercises followed by some tips on proof writing.

Overview of lecture so far

Vectors, matrices in $\mathbb{R}^n$
- vector addition, scalar multiplication
- matrix-vector and matrix-matrix multiplication
- norm, dot product
- different types of matrices (symmetric, triangular, ...)
- linear combinations, span

linear independence
- rank
- the inverse theorem, inverses
- CR decomposition

linear systems of equations
- Gauss-Elimination
- LU decomposition
- rref
- general solution of SLE

vector spaces
- subspaces
- bases
- dimension

orthogonality
- orthogonal subspaces
- orthogonal complement
- the four fundamental subspaces: $C(A)$, $N(A)$, $C(A^T)$, $N(A^T)$

LinAlg Recap Exercise

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1 LU decomposition

Consider the following matrix $A \in \mathbb{R}^{3\times 3}$ where $p \in \mathbb{R}$:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 2 & p & p \end{pmatrix}$$

1.1

Write down elimination matrices E_{21}, E_{31}, and E_{32} that introduce zeros in the $(2,1)$ $(3,1)$, and $(3,2)$ entries so that $E_{32}E_{31}E_{21}A = U$ is upper triangular. Their entries may depend on p.

subtract two times first row from second

I recommend to apply Gauss Elimination as normal and getting these on the way

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \qquad E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & p & 1 \end{pmatrix}$$

1.2

Write down the lower and upper triangular factors L and U that multiply to make $A = LU$. The triangular factors may depend on the parameter p.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -p & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & (-p-2) \end{pmatrix}$$

$E_{32}E_{31}E_{21}A = U$

$= (E_{32}E_{31}E_{21})^{-1}$
$= E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$

1.3

Why is A not invertible if $p = -2$?

Because then $U = rref(A) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ which only has rank 2 (two pivots, not full rank) and is hence not invertible.

1.4

If $p \neq -2$, $\quad \text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ — We get this either by computing it or using the fact A is invertible $\iff \text{rref}(A) = I$

1.5

Let $p = -2$. Find rref(A) and bases for $N(A)$ and $C(A)$.

1. $\text{rref}(A) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ — eliminate further (as much as possible, → just flipping sign of second row; we call this Gauss-Jordan Elimination)
2. A basis for $C(A)$ is given by: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \right\}$ — columns of A with pivot in ref(A) = U or rref(A) above
3. A basis for $N(A)$ is given by: $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$ — solve $Ax = 0$, get solution vector in terms of free variables and then get one vector per free variable containing the respective coefficients ("separating" the free variables) → see week 6

2 References

Exercises 1.1–1.3: https://github.com/mitmath/1806/blob/master/exams/exam1.pdf

LinAlg Recap weeks 1-7

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1 True/false and open questions

For the following questions let $x \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{m\times n}$ and V be a vector space.

The general/abstract def. of a norm contains these

1. Why can $||x||$ never be negative? Possible explanations: via def. of euclidean norm, fact that $||x||$ is the length of x and only the zero vector has length 0.
2. $||x|| = 0$ if and only if $x = 0$ (the zero vector)
3. If A is invertible, rank(A) = n, full rank
4. When is $U \subseteq V$ a subspace of V? When U is also a vector space. It suffices to show the following: 1. $0 \in U$ (U is nonempty) 2. For any $u_1, u_2 \in U$, $\alpha \in \mathbb{F}$: $u_1 + u_2 \in U$, $\alpha u_1 \in U$
5. We can compute the $A = CR$ decomposition with the Gauss-Jordan algorithm (to compute $rref(A)$) True! See week 7.
6. Consider B: The number of linearly independent rows does not always equal the number of linearly independent columns. False! $\text{rank}(A) = \text{rank}(A^T)$
7. How would you prove a set of vectors $B \subseteq V$ is a basis of V? Per def.: 1. span $B = V$ 2. B is linearly independent
8. If any vector $v \in \text{span}(v_1, \dots, v_n)$ can be uniquely expressed as a linear combination of $v_1, \dots v_n$, we call $v_1, \dots, v_n$ linearly independent → see week 2
9. A basis for the set of polynomials with real coefficients of degree less than or equal to 3 is given by $\{1, x, x^2, x^3\}$
10. Let $\mathbf{B}$ be a basis of V and $\mathbf{C}$ be a generating set of V (span$(\mathbf{C}) = V$). How do $\mathbf{B}$ and $\mathbf{C}$ differ? C doesn't have to be linearly independent, $|B| \leq |C|$
11. Multiplying A with elimination matrices from the left doesn't change the span of rows and span of columns of A False! It doesn't change span of rows, linear independence of columns but span of columns usually changes!
12. If dim$N(A) > 0$ we know that $Ax = b$ does not have a unique solution True! We don't know if there is a solution
13. How can you compute A^{-1} (assuming it exists)? $[A \mid I] \rightsquigarrow [I \mid A]$ → see week 4
14. $C(B)$ is a subspace of $\mathbb{R}^n$ False! $C(B) \subseteq \mathbb{R}^m$ → see week 5
15. What can we say about A if $A^4 = I$? What kind of matrix could A be? Possible answers: $A = I$, A is rotation matrix that rotates one plane in $\mathbb{R}^n$ by 90° or 180°
16. All bases of subspaces of V have the same number of vectors

False! This only holds for the subspace $U = V$. Always keep in mind that a basis of U may only contain elements of U

Some tips on proof writing

- Make sure you understand the claim you are asked to prove as well as possible
  → knowing/looking up the precise definitions is important

- Have you done a similar proof before? If yes, often the same ideas/techniques can be applied.
  How much you have already been exposed to the material can play a huge role in how quickly you can solve exercises. For me personally, the first few proof exercises of a new topic usually take significantly longer to complete. Hang in there! One can learn a lot through the process of trying to come up with a proof. If you are stuck, coming back to a problem after a break can work wonders.

- How can the given conditions be used? Often, you are asked to prove implications of the form $A \Rightarrow B$.
  In almost all cases there is a reason you may assume A and prove B only under these assumptions. This might hint at how your proof should look like. If you haven't used some assumptions (yet), think how you could use them (or why they are not required, though this is unlikely).

- Write your proof in a way that a fellow student, including me (or a computer), could understand it just by reading your submission.
  Often using words can play a great role in increasing the readability of a proof.
  → define all objects you use, justify the steps and write short conclusions